

# STINESPRING TYPE THEOREM FOR A FINITE FAMILY OF MAPS ON HILBERT $C^*$ -MODULES

M. PLIEV

**ABSTRACT.** The aim of this article is to extend the results of Asadi M.B, B.V.R. Bhat, G. Ramesh, K. Sumesh about completely positive maps on Hilbert  $C^*$ -modules. We prove a Stinespring type theorem for a finite family of completely positive maps on Hilbert  $C^*$ -modules. We also show that any two minimal Stinespring representations are unitarily equivalent.

## 1. INTRODUCTION

Stinespring representation theorem is a fundamental theorem in the theory of completely positive maps. The study of completely positive maps is motivated by applications of the theory of completely positive maps to quantum information theory, where operator valued completely positive maps on  $C^*$ -algebras are used as a mathematical model for quantum operations, and quantum probability. A completely positive map  $\varphi : A \rightarrow B$  of  $C^*$ -algebras is a linear map with the property that  $[\varphi(a_{ij})]_{i,j=1}^n$  is a positive element in the  $C^*$ -algebra  $M_n(B)$  of all  $n \times n$  matrices with entries in  $B$  for all positive matrices  $[(a_{ij})]_{i,j=1}^n$  in  $M_n(A)$ ,  $n \in \mathbb{N}$ . Stinespring [13] shown that a completely positive map  $\varphi : A \rightarrow L(H)$  is of the form  $\varphi(\cdot) = S^* \pi(\cdot) S$ , where  $\pi$  is a  $\star$ -representation of  $A$  on a Hilbert space  $K$  and  $S$  is a bounded linear operator from  $H$  to  $K$ . Theorem about the structure of  $n \times n$  matrices whose entries are linear positive maps from  $C^*$ -algebra  $A$  to  $L(H)$ , known as completely  $n$ -positive linear maps, were obtained by Heo [3]. Hilbert  $C^*$ -modules are generalizations of Hilbert spaces and  $C^*$ -algebras. In [1] Asadi had considered a version of the Stinespring theorem for completely positive map on Hilbert  $C^*$ -modules. Later Bhat, Ramesh and Sumesh in [2] had removed some technical conditions. Skiede in [13] had considered whole construction in a framework of the  $C^*$ -correspondences. Finally Joita in [5, 6] had proved covariant version of the Stinespring theorem and Radon-Nikodym theorem. In this paper, we shall prove a version of the Stinespring theorem for a finite families of the completely positive maps on Hilbert  $C^*$ -modules.

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## 2. PRELIMINARIES

The goal of this section is to introduce some basic definitions and facts. General information on  $C^*$ -algebras, Hilbert  $C^*$ -modules and completely positive maps the reader can find in the books [7, 8, 9, 10, 11].

We denote Hilbert spaces by  $H_1, H_2, K_1, K_2$  etc and the corresponding inner product and the induced norm by  $\langle \cdot, \cdot \rangle$  and  $\|\cdot\|$  respectively. Throughout we assume that the inner product is conjugate linear in the first variable and linear in the second variable. The space of bounded linear operators from  $H_1$  to  $H_2$  is denoted by  $L(H_1, H_2)$  and  $L(H_1) := L(H_1, H_1)$ . We denote  $C^*$ -algebras by  $A, B$  etc. The  $C^*$ -algebra of all  $n \times n$  matrices with entries from  $A$  is denoted by  $M_n(A)$ .

A Hilbert  $C^*$ -module  $V$  over  $C^*$ -algebra  $A$  is a linear space which is also a right  $A$ -module, equipped with an  $A$ -valued inner product  $\langle \cdot, \cdot \rangle_A$  that is  $V$  is  $\mathbb{C}$ -linear and  $A$ -linear in the second variable and conjugate linear in the first variable such that  $V$  is complete with the norm  $\|x\| = \|\langle x, x \rangle_A\|_A^{\frac{1}{2}}$ .  $V$  is full if the closed bilateral  $\star$ -sided ideal  $\langle V, V \rangle_A$  of  $A$  generated by  $\{\langle x, y \rangle_A : x, y \in V\}$  coincides with  $A$ . Remind the reader that  $L(H_1, H_2)$  is a Hilbert  $L(H_1)$ -module for any two Hilbert spaces  $H_1, H_2$ , with the following operations:

$$(2.1) \quad \text{module map } : (T, S) \mapsto TS : L(H_1, H_2) \times L(H_1) \rightarrow L(H_1, H_2);$$

$$(2.2) \quad \text{inner product } \langle T, S \rangle \mapsto T^*S : L(H_1, H_2) \times L(H_1, H_2) \rightarrow L(H_1).$$

A representation of  $V$  on the Hilbert spaces  $H_1$  and  $H_2$  is a map  $\Psi : V \rightarrow L(H_1, H_2)$  with the property that there is a  $\star$ -representation  $\pi$  of  $A$  on the Hilbert space  $H_1$  such that

$$\langle \Psi(x), \Psi(y) \rangle = \pi(\langle x, y \rangle)$$

for all  $x, y \in V$ . If  $V$  is full, then the  $\star$ -representation  $\pi$  associated to  $\Psi$  is unique. A representation  $\Psi : V \rightarrow L(H_1, H_2)$  of  $V$  is nondegenerate if  $[\Psi(V)(H_1)] = H_2$  and  $[\Psi(V)^*(H_2)] = H_1$  (here,  $[Y]$  denotes the closed subspace of a Hilbert space  $Z$  generated by subset  $Y \subset Z$ ). A map  $\Phi : V \rightarrow L(H_1, H_2)$  is called *completely positive* on  $V$  if there is a linear completely positive map  $\varphi : A \rightarrow L(H_1)$  such that

$$\langle \Phi(x), \Phi(y) \rangle = \varphi(\langle x, y \rangle)$$

for all  $x, y \in V$ .

Linear map  $\varphi : A \rightarrow B$  is said to be positive if  $\varphi(a^*a) \geq 0$ , for all  $a \in A$ . An  $n \times n$  matrix  $[\varphi_{ij}]_{i,j=1}^n$  of linear maps from  $A$  to  $B$  can be regarded as a linear map  $[\varphi] : M_n(A) \rightarrow M_n(B)$  defined by

$$[\varphi]([a_{ij}]_{i,j=1}^n) = [\varphi_{ij}(a_{ij})]_{i,j=1}^n$$

We say that  $[\varphi]$  is a completely  $n$ -positive linear map from  $A$  to  $B$  if  $[\varphi]$  is a completely positive linear map from  $M_n(A)$  to  $M_n(B)$ . If  $[\varphi_{ij}]_{i,j=1}^n$  is a completely  $n$ -positive linear map from  $A$  to  $B$ , then  $\varphi_{ii}$  is a completely positive linear map from  $A$  to  $B$  for each  $i \in \{1, \dots, n\}$ .

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### 3. MAIN RESULT

In this section we strengthen Bhat, Ramesh and Sumesh theorem and discuss the minimality of the representation.

Let  $V$  be a Hilbert  $C^*$ -module over  $A$  and let  $H_1, H_2$  be Hilbert spaces. Let  $\Phi_i, i \in \{1, \dots, n\}$  be a maps  $\Phi_i : V \rightarrow L(H_1, H_2)$ .

**Definition 3.1.** A  $n$ -tuple of maps  $\Phi = (\Phi_1, \dots, \Phi_n)$  is called *completely positive*, if there is a completely  $n$ -positive map  $[\varphi]$  from  $A$  to  $L(H_1)$  such that

$$[\langle \Phi_i(x), \Phi_j(y) \rangle]_{i,j=1}^n = [\varphi_{ij} \langle x, y \rangle]_{i,j=1}^n$$

for every  $x, y \in V$ .

**Remark 3.2.** It is obvious that every map  $\Phi_i, i \in \{1, \dots, n\}$  is completely positive.

**Theorem 3.3.** Let  $A$  be a unital  $C^*$ -algebra,  $V$  be a Hilbert  $A$ -module,  $[\varphi_{ij}]_{i,j=1}^n : A \rightarrow L(H_1)$  be a  $n$ -completely positive map and let  $\Phi = (\Phi_1, \dots, \Phi_n)$ ,  $\Phi_i : V \rightarrow L(H_1, H_2), i \in \{1, \dots, n\}$  be a  $[\varphi]$ -completely positive  $n$ -tuple. Then there exists a data  $(\pi, S_1, \dots, S_n, K_1), (\Psi, W_1, \dots, W_n, K_2)$ , where

- (1)  $K_1$  and  $K_2$  are Hilbert spaces;
- (2)  $\Psi : V \rightarrow L(K_1, K_2)$  is a representation of  $V$  on the Hilbert spaces  $K_1$  and  $K_2$ ,  $\pi : A \rightarrow L(K_1)$  is a unital  $\star$ -homomorphism associated with  $\Psi$ ,  $S_i : H_1 \rightarrow K_1$  are isometric linear operators,  $W_i : H_2 \rightarrow K_2$  are coisometric linear operators for every  $i \in \{1, \dots, n\}$ , such that

$$\varphi_{ij}(a) = S_i^* \pi_A(a) S_j \text{ for all } a \in A; i, j \in \{1, \dots, n\}$$

and

$$\Phi_i(x) = W_i^* \Psi(x) S_i \text{ for all } x \in V \text{ and every } i \in \{1, \dots, n\}.$$

*Proof.* At first we prove existence of  $\pi, K_1$  and  $S_1, \dots, S_n$ . The more general construction like that is known in the literature (see for example [4], Theorem 4.1.8), but for sake of a completeness we shall consider it here. We denote by  $(A \otimes_{\text{alg}} H_1)^n$  the direct sum of  $n$  copies of the algebraic tensor product  $A \otimes H_1$ .  $(A \otimes_{\text{alg}} H_1)^n$  is a vector space with a map  $\langle \cdot, \cdot \rangle_0 : (A \otimes_{\text{alg}} H_1)^n \times (A \otimes_{\text{alg}} H_1)^n$  defined by formula

$$\left\langle \sum_{s=1}^m (a_{is} \otimes \xi_{is})_{i=1}^n, \sum_{t=1}^l (b_{jt} \otimes \eta_{jt})_{j=1}^n \right\rangle_0 = \sum_{s,t=1}^{m,l} \sum_{i,j=1}^n \langle \xi_{is}, \varphi_{ij}(a_{is}^* b_{jt}) \eta_{jt} \rangle$$

is  $\mathbb{C}$ -linear in its second variable. It is not difficult to check that

$$\left( \left\langle \sum_{s=1}^m (a_{is} \otimes \xi_{is})_{i=1}^n, \sum_{t=1}^l (b_{jt} \otimes \eta_{jt})_{j=1}^n \right\rangle_0 \right)^* = \left\langle \sum_{t=1}^l (b_{jt} \otimes \eta_{jt})_{j=1}^n, \sum_{s=1}^m (a_{is} \otimes \xi_{is})_{i=1}^n \right\rangle_0$$

for all  $(a_{is} \otimes \xi_{is})_{i=1}^n, (b_{jt} \otimes \eta_{jt})_{j=1}^n \in (A \otimes_{\text{alg}} H_1)^n$  and

$$\left\langle \sum_{s=1}^m (a_{is} \otimes \xi_{is})_{i=1}^n, \sum_{s=1}^m (a_{is} \otimes \xi_{is})_{i=1}^n \right\rangle_0 \geq 0$$

Let  $M := \{\zeta : (A \otimes_{\text{alg}} H_1)^n; \langle \zeta, \zeta \rangle_0 = 0\}$ .  $M$  is a subspace of  $(A \otimes_{\text{alg}} H_1)^n$ . Then by Cauchy-Schwarz Inequality,  $M$  is a subspace of  $(A \otimes_{\text{alg}} H_1)^n$ . Then  $(A \otimes_{\text{alg}} H_1)^n/M$  becomes pre-Hilbert space with inner product defined by

$$\langle \zeta_1 + M, \zeta_2 + M \rangle := \langle \zeta_1, \zeta_2 \rangle_0.$$

The completion of  $(A \otimes_{\text{alg}} H_1)^n/M$  with respect to the topology induced by the inner product is denoted by  $K_1$ . We denote by  $\xi_i$  the element in  $(A \otimes H_1)^n$  whose  $i^{\text{th}}$  component is  $1 \otimes \xi$  and all other component are 0. Now we can define a map  $S_i : H_1 \rightarrow K_1$  by

$$S_i(\xi) = \xi_i + M$$

Let denote by  $\xi_{a,i}$  the element in  $(A \otimes_{\text{alg}} H_1)^n/M$  whose  $i^{\text{th}}$  component is  $a \otimes \xi$  and all other component are 0. Let  $a \in A$ . Consider the linear map  $\pi(a) : (A \otimes_{\text{alg}} H_1)^n \rightarrow (A \otimes_{\text{alg}} H_1)^n$  defined by

$$\pi(a)(a_i \otimes \xi_i)_{i=1}^n = (aa_i \otimes \xi_i)_{i=1}^n.$$

Linear map  $\pi(a)$  can be extended by linearity and continuity to a linear map, denoted also by  $\pi(a)$ , from  $K_1$  to  $K_1$ . The fact that  $\pi(a)$  is a representation of  $A$  on  $L(K_1)$  it is showed in the same manner as in the proof of Theorem 3.3.2 from [4]. It is not difficult to check that  $\pi(a_i)S_i\xi_i = \xi_{i,a} + M$ . Therefore the subspace of  $K_1$  generated by  $\pi(a_i)S_i\xi_i$ ,  $i \in \{1, \dots, n\}$ ,  $\xi_i \in H_1$ ,  $a_i \in A$  is exactly  $(A \otimes_{\text{alg}} H_1)^n/M$ .

Let  $K_2 := [\{\Psi(V)S_i(H_1), i = 1, \dots, n\}]$ . Now we can define  $\Psi : V \rightarrow L(K_1, K_2)$  as follows:

$$\Psi(x) \left( \sum_{i=1}^n \sum_{s=1}^m \pi(a_{is}) S_i \xi_{is} \right) := \sum_{i=1}^n \sum_{s=1}^m \Phi_i(xa_{is}) \xi_{is},$$

where  $x \in V$ ,  $a_{is} \in A$ ,  $\xi_{is} \in H_1$ ,  $1 \leq i \leq n, 1 \leq s \leq m$ ,  $m \in \mathbb{N}$ . We claim that  $\Psi(x)$  is well defined

$$\begin{aligned} \left\| \Psi(x) \left( \sum_{i=1}^n \sum_{s=1}^m \pi(a_{is}) S_i \xi_{is} \right) \right\|^2 &= \left\| \sum_{i=1}^n \sum_{s=1}^m \Phi_i(xa_{is}) \xi_{is} \right\|^2 = \\ &= \left\langle \sum_{s=1}^m \sum_{i=1}^n \Phi_i(xa_{is}) \xi_{is}, \sum_{r=1}^m \sum_{j=1}^n \Phi_j(xa_{jr}) \xi_{jr} \right\rangle = \\ &= \sum_{s,r=1}^m \sum_{i,j=1}^n \langle \xi_{is}, \Phi_i(xa_{is})^* \Phi_j(xa_{jr}) \xi_{jr} \rangle = \\ &= \sum_{s,r=1}^m \sum_{i,j=1}^n \langle \xi_{is}, \varphi_{ij}(\langle xa_{is}, xa_{jr} \rangle) \xi_{jr} \rangle = \end{aligned}$$

$$\begin{aligned}
&= \sum_{s,r=1}^m \sum_{i,j=1}^n \langle \xi_{is}, S_i^* \pi(a_{is}^* \langle x, x \rangle a_{jr}) S_j \xi_{jr} \rangle = \\
&= \sum_{s,r=1}^m \sum_{i,j=1}^n \langle \pi(a_{is}) S_i(\xi_{is}), \pi(\langle x, x \rangle) \pi(a_{jr}) S_j \xi_{jr} \rangle = \\
&= \left\langle \sum_{s=1}^m \sum_{i=1}^n \pi(a_{is}) S_i(\xi_{is}), \pi(\langle x, x \rangle) \left( \sum_{r=1}^m \sum_{j=1}^n \pi(a_{jr}) S_j \xi_{jr} \right) \right\rangle \leq \\
&\leq \left\| \pi(\langle x, x \rangle) \right\| \left\| \left( \sum_{r=1}^m \sum_{i=1}^n \pi(a_{i,r}) S_i \xi_{i,r} \right) \right\|^2 \leq \\
&\leq \|x\|^2 \left\| \left( \sum_{r=1}^m \sum_{i=1}^n \pi(a_{i,r}) S_i \xi_{i,r} \right) \right\|^2.
\end{aligned}$$

Hence  $\Psi(x)$  is well defined and bounded. Hence it can be extended to the whole of  $K_1$ . Now we prove that  $\Psi$  is a representation. For this let  $x, y \in V$ ;  $a_{is}, b_{jr} \in A$ ;  $\xi_{is}, \eta_{jr} \in H_1$ ;  $1 \leq i, j \leq n$ ;  $1 \leq s \leq l$ ,  $1 \leq r \leq m$ ;  $n, m \in \mathbb{N}$ . Then we have

$$\begin{aligned}
&\left\langle \Psi(x)^* \Psi(y) \left( \sum_{r=1}^m \sum_{j=1}^n \pi(b_{j,r}) S_j \eta_{j,r} \right), \sum_{s=1}^l \sum_{i=1}^n \pi(a_{i,s}) S_i \xi_{i,s} \right\rangle = \\
&= \left\langle \sum_{r=1}^m \sum_{j=1}^n \Phi_j(y b_{j,r}) \eta_{j,r}, \sum_{s=1}^l \sum_{i=1}^n \Phi_i(x a_{i,s}) \xi_{i,s} \right\rangle = \\
&= \sum_{s=1}^l \sum_{r=1}^m \sum_{i,j=1}^n \langle \Phi_i(x a_{i,s})^* \Phi_j(y b_{j,r}) \eta_{j,r}, \xi_{i,s} \rangle = \\
&= \sum_{s=1}^l \sum_{r=1}^m \sum_{i,j=1}^n \langle \varphi_{ij}(\langle x a_{i,s}, y b_{j,r} \rangle) \eta_{j,r}, \xi_{i,s} \rangle = \\
&= \sum_{s=1}^l \sum_{r=1}^m \sum_{i,j=1}^n \langle S_i^* \pi(a_{is}^* \langle x, y \rangle a_{jr}) S_j \eta_{j,r}, \xi_{i,s} \rangle = \\
&= \left\langle \pi(\langle x, y \rangle) \left( \sum_{r=1}^m \sum_{j=1}^n \pi(b_{j,r}) S_j \eta_{j,r} \right), \sum_{s=1}^l \sum_{i=1}^n \pi(a_{i,s}) S_i \xi_{i,s} \right\rangle
\end{aligned}$$

Thus  $\Psi(x)^* \Psi(y) = \pi(\langle x, y \rangle)$  on the dense set and hence they are equal on  $K_1$ . Note  $K_2 \subset H_2$ . Denote subspace  $[\Phi_i(V)(H_1)]$  of the  $H_2$  by  $K_{2i}$ . Let  $W_i := P_{K_{2i}}$ ,  $i \in \{1, \dots, n\}$  the orthogonal projection from  $H_2$  to  $K_{2i}$ . Then  $W_i^* : K_{2i} \rightarrow H_2$  is a inclusion map. Hence  $W_i W_i^* = I_{K_{2i}}$  for every  $i \in \{1, \dots, n\}$ . Now we give a representation for  $\Phi$ . For every  $x \in V$  and  $\xi \in H_1$ , we have

$$\Phi_i(x)(\xi) = W_i^* \Psi(x) S_i(\xi) \text{ for every } i \in \{1, \dots, n\}.$$

□

**Definition 3.4.** Let  $[\varphi]$  and  $\Phi$  be as an Theorem 3.3. We say that a data  $(\pi, S_1, \dots, S_n, K_1)$ ,  $(\Psi, W_1, \dots, W_n, K_2)$  is a *Stinespring representation* of  $(\varphi, \Phi)$  if conditions (1)–(2) of Theorem 3.3 is satisfied. Such a representation is said to be *minimal* if

- 1)  $K_1 = [\{\pi(A)S_i(H_1); i = 1, \dots, n\}]$ ;
- 2)  $K_2 = [\{\Psi(V)S_i(H_1); i = 1, \dots, n\}]$ .

**Theorem 3.5.** Let  $[\varphi]$  and  $\Phi$  be as an Theorem 3.3. Assume that  $(\pi, S_1, \dots, S_n, K_1)$ ,  $(\Psi, W_1, \dots, W_n, K_2)$  and  $(\pi', S'_1, \dots, S'_n, K'_1)$ ,  $(\Psi', W'_1, \dots, W'_n, K'_2)$  are *minimal Stinespring representations*. Then there exists unitary operators  $U_1 : K_1 \rightarrow K'_1$ ,  $U_2 : K_2 \rightarrow K'_2$  such that

- (1)  $U_1 S_i = S'_i, \forall i \in \{1, \dots, n\}; U_1 \pi(a) = \pi'(a)U_1, \forall a \in A$ .
- (2)  $U_2 W_i = W'_i, \forall i \in \{1, \dots, n\}; U_2 \Psi(x) = \Psi'(x)U_2, \forall x \in V$ .

That is a following diagram commutes, for all  $a \in A, x \in V, i \in \{1, \dots, n\}$

$$\begin{array}{ccccccccc} H_1 & \xrightarrow{S_i} & K_1 & \xrightarrow{\pi(a)} & K_1 & \xrightarrow{\Psi(x)} & K_2 & \xleftarrow{W_i} & H_2 \\ \downarrow Id & & \downarrow U_1 & & \downarrow U_1 & & \downarrow U_2 & & \downarrow Id \\ H_1 & \xrightarrow{S'_i} & K'_1 & \xrightarrow{\pi'(a)} & K'_1 & \xrightarrow{\Psi'(x)} & K'_2 & \xleftarrow{W'_i} & H_2 \end{array}$$

*Proof.* Let us prove the existence of the unitary map  $U_1 : H_1 \rightarrow K_1$ . First define  $U_1$  on the dense subspace — linear span  $\{\pi(A)S_i(H_1); i = 1, \dots, n\}$ .

$$U_1 \left( \sum_{s=1}^m \sum_{i=1}^n \pi(a_{is}) S_i(\xi_{is}) \right) := \left( \sum_{s=1}^m \sum_{i=1}^n \pi'(a_{is}) S'_i(\xi_{is}) \right)$$

where  $a_{is} \in A, \xi_{is} \in H_1, m \in \mathbb{N}$ . It is not difficult to check that  $U_1$  is an onto isometry. Denote the extension of  $U_1$  to  $K_1$  by  $U_1$  itself. Then  $U_1$  is unitary and satisfies the condition in (1). Now define  $U_2$  on the dense subspace — linear span  $\{\Psi(V)S_i(H_1); i = 1, \dots, n\}$ .

$$U_2 \left( \sum_{i=1}^n \sum_{s=1}^m \Psi(x_{is}) S_i \xi_{is} \right) := \left( \sum_{i=1}^n \sum_{s=1}^m \Psi'(x_{is}) S'_i \xi_{is} \right),$$

where  $x_{is} \in V, \xi_{is} \in H_1, m \in \mathbb{N}$ . Using the fact that  $S_i, S'_i$  are isometric operators for every  $i \in \{1, \dots, n\}$  we have

$$U_2 \left( \sum_{s=1}^m \Psi(x_{is}) S_i \xi_{ns} \right) = \sum_{s=1}^m \Psi'(x_{is}) S'_i \xi_{ns},$$

and so  $U_2(K_{2i}) = K'_{2i}$ , where  $K_{2i} = [\Psi(V)S_i(H_1)]$  and  $K'_{2i} = [\Psi'(V)S'_i(H_1)]$ . We can see that  $U_2$  is well defined and can be extended to a unitary map. For this consider

$$\left\| \left( \sum_{i=1}^n \sum_{s=1}^m \Psi'(x_{is}) S'_i \xi_{is} \right) \right\|^2 = \left\langle \sum_{i=1}^n \sum_{s=1}^m \Psi'(x_{is}) S'_i \xi_{is}, \sum_{j=1}^n \sum_{r=1}^m \Psi'(x_{jr}) S'_j \xi_{jr} \right\rangle =$$

$$\begin{aligned}
&= \sum_{s,r=1}^m \sum_{i,j=1}^n \langle \Psi'(x_{is}) S'_i \xi_{is}, \Psi'(x_{jr}) S'_j \xi_{jr} \rangle = \\
&= \sum_{s,r=1}^m \sum_{i,j=1}^n \langle \xi_{is}, S_i'^* \pi'(\langle x_{is}, x_{jr} \rangle) S'_j(\xi_{jr}) \rangle = \\
&= \sum_{s,r=1}^m \sum_{i,j=1}^n \langle \xi_{is}, \varphi_{ij}(\langle x_{is}, x_{jr} \rangle)(\xi_{jr}) \rangle = \\
&= \sum_{s,r=1}^m \sum_{i,j=1}^n \langle \xi_{is}, S_i^* \pi(\langle x_{is}, x_{jr} \rangle) S_j(\xi_{jr}) \rangle = \\
&= \sum_{s,r=1}^m \sum_{i,j=1}^n \langle \Psi(x_{is}) S_i \xi_{is}, \Psi(x_{jr}) S_j \xi_{jr} \rangle = \\
&= \left\langle \sum_{i=1}^n \sum_{s=1}^m \Psi(x_{is}) S_i \xi_{is}, \sum_{j=1}^n \sum_{r=1}^m \Psi(x_{jr}) S_j \xi_{jr} \right\rangle = \\
&\quad \left\| \sum_{i=1}^n \sum_{s=1}^m \Psi(x_{is}) S_i \xi_{is} \right\|^2.
\end{aligned}$$

Hence  $U_2$  is well defined and isometry, therefore  $U_2$  can be extended to whole of  $K_2$ . We call this extension  $U_2$  itself. Operator  $U_2$  is an onto isometry. We have noticed that  $(\pi, S_1, \dots, S_n, K_1)$ ,  $(\Psi, W_1, \dots, W_n, K_2)$  and  $(\pi', S'_1, \dots, S'_n, K'_1)$ ,  $(\Psi', W'_1, \dots, W'_n, K'_2)$  are Stinespring representations for  $([\varphi], \Phi)$ . Hence for every  $i \in \{1, \dots, n\}$  we have

$$\begin{aligned}
\Phi_i(x) &= W_i^* \Psi(x) S_i = W_i'^* \Psi'(x) S'_i = \\
&= W_i'^* U_2 \Psi(x) S_i.
\end{aligned}$$

Hence

$$(W_i^* - W_i'^* U_2) \Psi(x) S_i = 0 \Rightarrow$$

$$(W_i^* - W_i'^* U_2) \Psi(x) S_i(\xi) = 0, \forall x \in V, \xi \in H_1, i \in \{1, \dots, n\}.$$

Hence  $U_2 W_i = W_i'$  for every  $i \in \{1, \dots, n\}$ . Finally we show that  $U_2 \Psi(x) = \Psi'(x) U_1$  on the dense subspace

$$\left\{ \sum_{s=1}^m \sum_{i=1}^n \pi(a_{is}) S_i(\xi_{is}); a_{is} \in A, \xi_{is} \in H_1, m \in \mathbb{N} \right\}.$$

We must to recall that every representation  $\Psi : V \rightarrow L(K_1, K_2)$  has a property  $\Psi(xa) = \Psi(x)\pi(a)$  for every  $x \in V$  and  $a \in A$ . Then using the fact that  $\Psi$  and  $\Psi'$  are representations associated with  $\pi$  and  $\pi'$  respectively we have

$$U_2 \Psi(x) \left( \sum_{s=1}^m \sum_{i=1}^n \pi(a_{is}) S_i(\xi_{is}) \right) = U_2 \left( \sum_{s=1}^m \sum_{i=1}^n \Psi(xa_{is}) S_i \xi_{is} \right) =$$

$$\begin{aligned}
\sum_{s=1}^m \sum_{i=1}^n \Psi'(xa_{is}) S'_i \xi_{is} &= \Psi'(x) \left( \sum_{s=1}^m \sum_{i=1}^n \pi'(a_{is}) S'_i(\xi_{is}) \right) = \\
&= \Psi'(x) U_1 \left( \sum_{s=1}^m \sum_{i=1}^n \pi(a_{is}) S_i(\xi_{is}) \right).
\end{aligned}$$

□

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SOUTH MATHEMATICAL INSTITUTE OF THE RUSSIAN ACADEMY OF SCIENCES, STR.  
MARKUSA 22, VLADIKAVKAZ, 362027 RUSSIA